Fuzzy logic
Fuzzy approximate reasoning

3.class

Introduction

- uncertain processes
- dynamic engineering system models
- fundamental of the decision making in *fuzzy based real systems* is the **approximate reasoning**, which is a rule-based system

"Informally, by approximate or, equivalently, fuzzy reasoning, we mean the process or processes by which a possibly imprecise conclusion is deduced from a collection of imprecise premises. Such reasoning is, for the most part, qualitative rather than quantitative in nature and almost all of it falls outside of the domain of applicability of classical logic"

(Zadeh, L. A., (1979)).

- fuzzy systems are very much application-oriented
- systems need to be integrated into the well-known mathematical theories
- although application-oriented fuzzy systems seek to be simple and comprehensible, it is obvious that they are heavily related to the fields of classical multi-valued logic, operation research and functional analysis

Experts are confident in using special types of operations in fuzzy systems, such as

t-norms, *t-conorms*, *uninorms*,

and more generally,

aggregation operators,

and researchers are more and more meticulous in providing exact mathematical definitions for those .

1. Real operations used in fuzzy set theory

Real semiring

- ***t-norms and t-conorms***
- ***Aggregation operators***
- ***Uninorms and nullnorms***
- ***Many-valued logics***
- ***Lattice ordered monoids and left continuous uninorms and t-norms***

Real semiring

Let $[a,b]$ be a closed subinterval of $[-\infty,+\infty]$ (in some cases semi-closed subintervals will be considered) and let $\underline{\mathbf{p}}$ be a total order on $[a,b]$. A *semiring* is the structure $(\underline{\mathbf{p}},\oplus,\otimes)$ if the following hold:

- § $\oplus$ is pseudo-addition, i.e., a function $\oplus : [a,b] \times [a,b] \to [a,b]$ which is commutative, non-decreasing (with respect to $\underline{\mathbf{p}}$), associative and with a zero element denoted by $\mathbf{0}$;
- § $\otimes$ is pseudo-multiplication, i.e., a function $\otimes : [a,b] \times [a,b] \to [a,b]$ which is commutative, positively non-decreasing ($x \underline{\mathbf{p}} y$ implies $x \otimes z \underline{\mathbf{p}} x \otimes y$ where $z \in [a,b]_+ = \{z | z \in [a,b], \mathbf{0} \underline{\mathbf{p}} z\}$ associative and for which there exists a unit element denoted by $\mathbf{1}$.
- § $\mathbf{0} \otimes z = \mathbf{0}$

$$x \otimes (y \oplus z) = (x \otimes y) \oplus (x \otimes z)$$

t-norm

Definition 1.2.1. A function $T : [0,1]^2 \to [0,1]$ is called *triangular norm (t-norm)* if and only if it fulfills the following properties for all $x, y, z \in [0,1]$

(T1) $T(x, y) = T(y, x)$, i.e., the t-norm is commutative,

(T2) $T(T(x, y), z) = T(x, T(y, z))$, i.e., the t-norm is associative,

(T3) $x \leq y \Rightarrow T(x, z) \leq T(y, z)$, i.e., the t-norm is monotone,

(T4) $T(x,1) = x$, i.e., a neutral element exists, which is 1.

t-conorm

Definition 1.2.3. A function $S:[0,1]^2 \rightarrow [0,1]$ is called *triangular conorm (t-conorm)* if and only if it fulfills the following properties for all $x,y,z \in [0,1]$:

(S1) $S(x,y) = S(y,x)$, i.e., the t-conorm is commutative,

(S2) $S(S(x,y),z) = S(x,S(y,z))$, i.e., the t-conorm is associative,

(S3) $x \le y \Rightarrow S(x,z) \le S(y,z)$, i.e., the t-conorm is monotone,

(S4) $S(x,0) = x$, i.e., a neutral element exists, which is 0.

Duality

A function $S:[0,1]^2 \to [0,1]$ is a *t*-conorm if and only if there exists such a *t*-norm T that for all $(x,y)\in [0,1]^2$:

$$S(x,y)=1-T(1-x,1-y).$$

This duality allows us to translate many properties of *t*-norms into corresponding properties of *t*-conorms.

The basic t-norms and conorms

- $T_M(x,y) = min(x,y)$, the minimum t-norm,
- $T_D(x,y) = \begin{cases} 0 & if \quad (x,y) \in [0,1[^2 \\ 1 & otherwise \end{cases}$, the drastic product.
- $S_M(x,y) = max(x,y)$, the maximum *t*-conorm,
- $S_D(x,y) = \begin{cases} 1 & if \quad (x,y) \in]0,1]^2 \\ max(x,y) & otherwise \end{cases}$, the drastic sum.
- $T_D \leq T \leq T_M \leq S_M \leq S \leq S_D$.

Continuity

A t-norm T is *lower semi-continuous (upper semi-continuous)* if and only if it is left continuous (right continuous) in its first component, i.e. if for each $y \in [0,1]$ and for each sequence $(x_n)_{n \in N} \subset [0,1]^N$ we have, respectively,

$$\sup_{n \in N} T(x_n, y_n) = T\left(\sup_{n \in N} x_n, y_n \right),$$

$$\inf_{n \in N} T(x_n, y_n) = T\left(\inf_{n \in N} x_n, y_n \right).$$

Algebraic aspects

Let T be a t-norm.

(i) An element $a \in [0,1]$ is called an *idempotent element* of T if $T(a,a) = a$. Since 0 and 1 are idempotent elements for each t-norm T they are called *trivial idempotent elements* of T, and each idempotent element in $]0,1[$ will be called a *non-trivial idempotent element* of T.

(ii) An element $a \in [0,1]$ is called a *nilpotent element* of T if there exist some $n \in N$ such that $a_T^{(n)} = 0$.

(iii) An element $a \in [0,1]$ is called a *zero divisor* of T if there exist some $b \in]0,1[$ such that $T(a,b) = 0$.

For an arbitrary t-norm T we consider the following properties:

(i) The t-norm T is said to be *strictly monotone* if

$T(x,y) < T(x,z)$ whenever $x > 0$ and $y < z$.

(ii) The t-norm T is said to be *Archimedean* if

$(\forall (x,y) \in \,]0,1[)$ there exists an $n \in N$ such that $x_T^{(n)} < y$.

Representation theorems for basic operations

A t-norm T is continuous and Archimedian if and only if there exists a strictly decreasing and continuous function $t:[0,1]\rightarrow[0,+\infty]$ with $t(1)=0$ such that

$$T(x,y)=t^{(-1)}(t(x)+t(y))$$

where $t^{(-1)}$ is the pseudoinverse of *f* defined by

$$t^{(-1)}(x)=\begin{cases} t^{-1}(x) & if \quad x\le t(0) \\ 0 & otherwise \end{cases}.$$

Moreover, representation (1.1.) is unique up to a positive multiplicative constant.

Aggregation operators

An *aggregation operator* is a function $\mathbf{A} : \bigcup_{n \in N} [0,1]^n \to [0,1]$ such that:

(*i*) $\mathbf{A}(x_1, x_2, \ldots, x_n) \leq \mathbf{A}(y_1, y_2, \ldots, y_n)$ whenever $x_i \leq y_i$ for all $i \in \{1,2,\ldots,n\}$

(*ii*) $\mathbf{A}(x) = x$ for all $x \in [0,1]$

(*iii*) $\mathbf{A}(0,0,\ldots,0) = 0$ and $\mathbf{A}(1,1\ldots,1) = 1$.

$\mathbf{A}$ is an idempotent aggregation operator if and only if

$$T_M \leq \mathbf{A} \leq S_M .$$

Uninorms

A *uninorm* is a binary operation U on the unit interval, i.e., a function

$U:[0,1]^2 \to [0,1]$ which satisfies the following properties for all $x, y, z \in [0,1]$

(U1) $U(x,y) = U(y,x)$, i.e. the uninorm is commutative,

(U2) $U(U(x,y),z) = U(x,U(y,z))$, i.e. the uninorm is associative,

(U3) $x \le y \Rightarrow U(x,z) \le U(y,z)$, i.e. the uninorm monotone,

(U4) $U(e,x) = x$, **i.e., a neutral element exists, which is** $e \in [0,1]$**.**

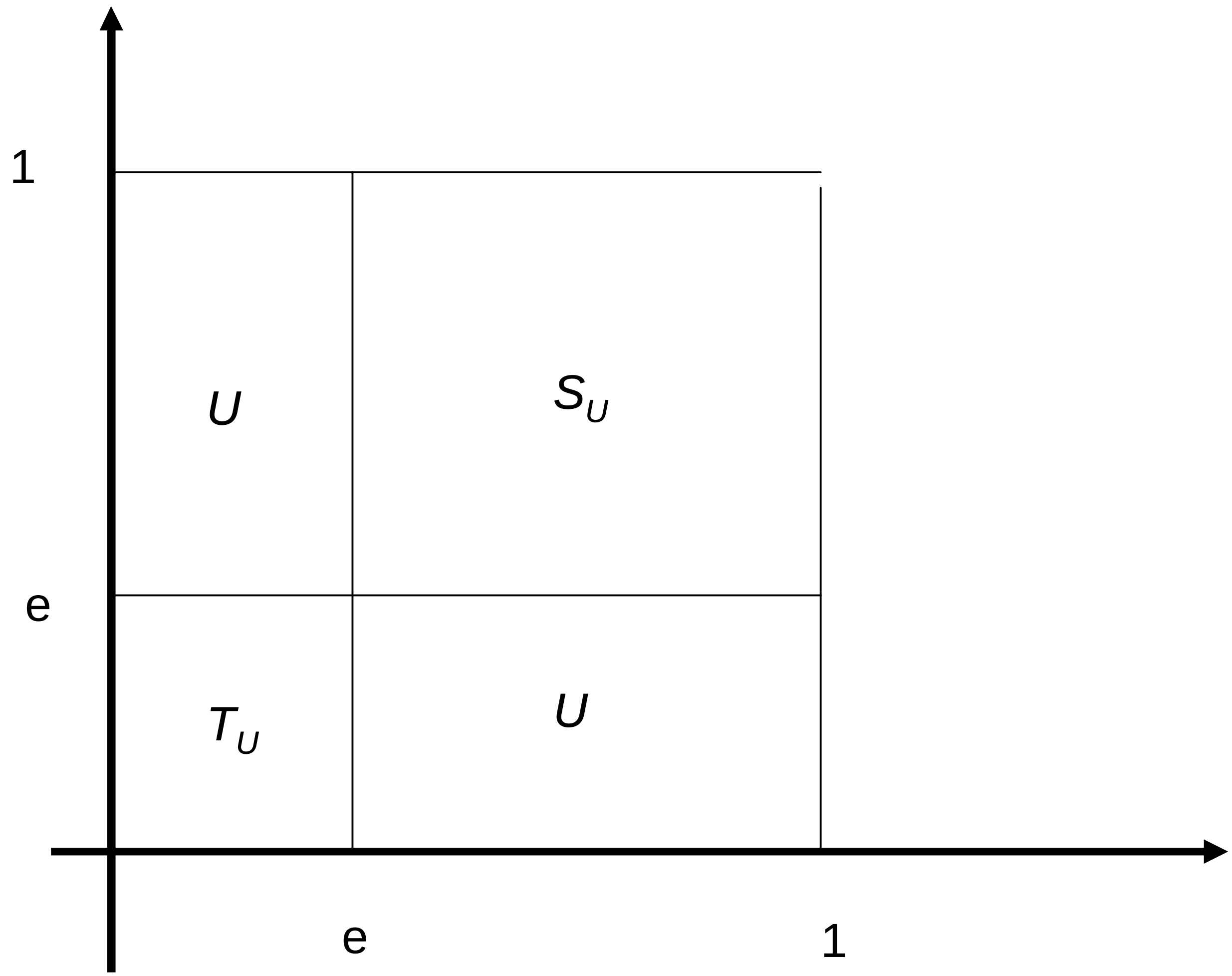
It is evident that, for an arbitrary uninorm U with the neutral element $e \in \left]0,1\right[$, the operations $T_U, S_U : [0,1]^2 \to [0,1]$, which are defined by

$$T_U(x,y) = \frac{1}{e} U(ex, ey),$$

$$S_U(x,y) = \frac{1}{1-e}\left(U\left(e + (1-e)x, e + (1-e)y\right) - e\right),$$

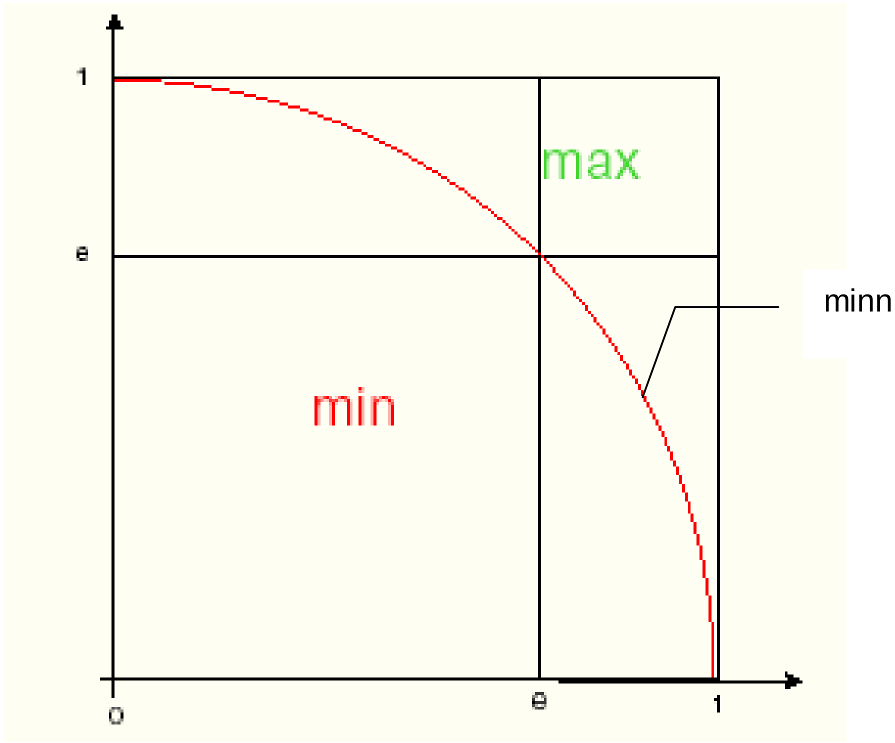
are t-norms and t-conorms, respectively

Structure of the uninorm



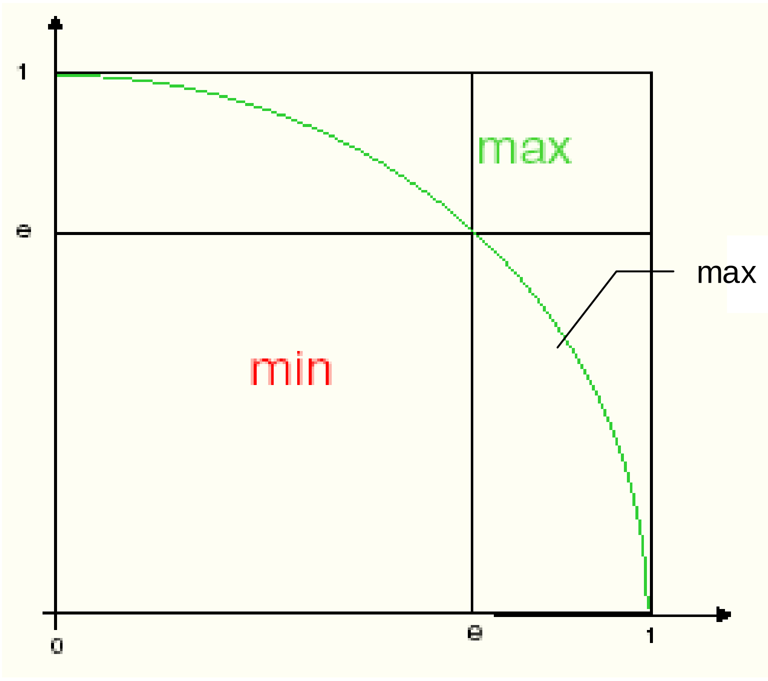
Representable uninorms

- The structure of the left-continuous conjunctive idempotent uninorm



$$U(0,1) = U(1,0) = 0$$

- The structure of the right-continuous disjunctive idempotent uninorm



$$U(0,1)=U(1,0)=1$$

If we suppose an unary operator g on set [0,1], then g is called

(i) sub-involutive if $g(g(x)) \leq x$ for $(\forall x \in [0,1])$, and

(ii) super-involutive if $g(g(x)) \geq x$ for $(\forall x \in [0,1])$.

A binary operator U is a conjunctive left-continuous idempotent uninorm with the neutral element $e \in]0,1]$ if and only if there exists a super-involutive decreasing unary operator g with the fixpoint e and $g(0) = 1$ such that U for any $\forall (x, y) \in [0,1]^2$ is given by

$$U(x,y) = \begin{cases} min(x,y) & if \quad y \leq g(x) \\ max(x,y) & elsewhere \end{cases}.$$

A binary operator U is a disjunctive right-continuous idempotent uninorm with the neutral element $e \in [0,1[$ if and only if there exists a sub-involutive decreasing unary operator g with the fixpoint e and $g(1) = 0$ such that U for any $\forall (x, y) \in [0,1]^2$ is given by

$$U(x, y) = \begin{cases} max(x, y) & if \quad y \geq g(x) \\ min(x, y) & elsewhere \end{cases}.$$

Many-valued logics

- Implications
- Definition
- Implications by t-norms, t-conorms and negations

The modeling “if … then …” rules with fuzzy predicates is based on fuzzy implications

In fuzzy logic, the basic theory of connectives *and*, *or*, *not* is well developed and their functional models (t-norms, t-conorms and strong negations) are widely accepted (Weber, S.,(1983)). However, there is no such clear, and, in some sense, unique way of defining fuzzy implications. (Fodor, J., (1996))

Implication

An implication is a function $I : [0,1]^2 \to [0,1]$ with following properties

(I1) if $x \le z$ then $I(x,y) \ge I(z,y)$ for $\forall y \in [0,1]$,

(I2) if $y \le t$ then $I(x,y) \le I(x,t)$ for $\forall x \in [0,1]$,

(I3) $I(0,x) = 1$ for $\forall x \in [0,1]$,

(I4) $I(x,1) = 1$ for $\forall x \in [0,1]$,

(I5) $I(1,0) = 0$.

R-implication

An *R*-implication associated with a t-norm *T* is defined by

$$I_T(x,y) = sup\{z | T(x,z) \le y\}$$

Theorem (Fodor and Roubens (1994))

Assume that *T* is a continuous Archimedean t-norm with additive generator *f*. *Then*

$$I_T(x,y) = f^{-1}(max\{f(y) - f(x), 0\}) =$$

$$\boldsymbol{I_T}(x,y) = \begin{cases} 1 & y > x \\ f^{(-1)}(f(y) - f(x)) & y \le x \end{cases}.$$

“T-implication”

A very important class of implications is *t-norm* implications group, defined by

$$I(x,y) = T(x,y).$$

Although these implication do not verify the properties of the implications they are used as a model in many applications in fuzzy logic, for example as Mamdani “implication”.

Implication? Relation?
Fuzzy relation

A function $R : X \times Y \rightarrow [0,1]$ is called (binary) fuzzy relation of ty (X, Y). The value $R(x, y)$ is interpreted as the degree to which $x \in X$ a $y \in Y$ are in relation . If $X = Y$ we say that *R* is a fuzzy relation on *X*.

The membership function of the composition of a fuzzy set *C* and fuzzy relation *R* is defined by

$$(C \circ R)(y) = \sup_{x \in X} T(C(x), R(x, y))$$

for all $y \in Y$.

Interpretation of connectives in fuzzy logic

- A many valued propositional logic in which the class of truth values is modelled by the unit interval [0,1], and which forms an extension of the classical Boolean logic, i.e., the two valued logic with truth values {0,1}, is quite often called a fuzzy logic (Gottwald (2001)). In such a logic, the conjunction is usually interpreted by a t-norm.

Residuum-based fuzzy logic

- All of conditions introduced in this section for the *R*-fuzzy logic *RT* and *R*-implication with continuous t-norm *T* are coherent with the conditions for *R*-implication.
- In following investigations there is always strictly determined, is it a continuous or left continuous, Archimedian or general t-norm used by approximate reasoning.

Implication

An implication is a function $I:[0,1]^2 \to [0,1]$ with following properties

(I1) if $x \le z$ then $I(x,y) \ge I(z,y)$ for $\forall y \in [0,1]$,

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$$I_T(x,y) = f^{-1}(max\{f(y) - f(x),0\}) =$$

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A function $R: X \times Y \rightarrow [0,1]$ is called (binary) fuzzy relation of ty (X, Y). The value $R(x, y)$ is interpreted as the degree to which $x \in X$ a $y \in Y$ are in relation . If $X = Y$ we say that R is a fuzzy relation on X.

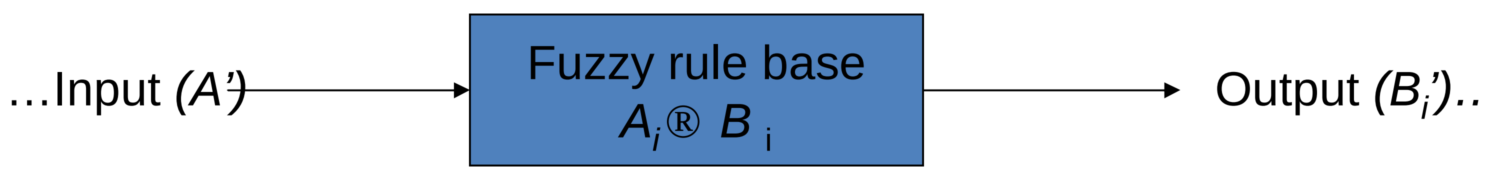
The membership function of the composition of a fuzzy set C and fuzzy relation R is defined by

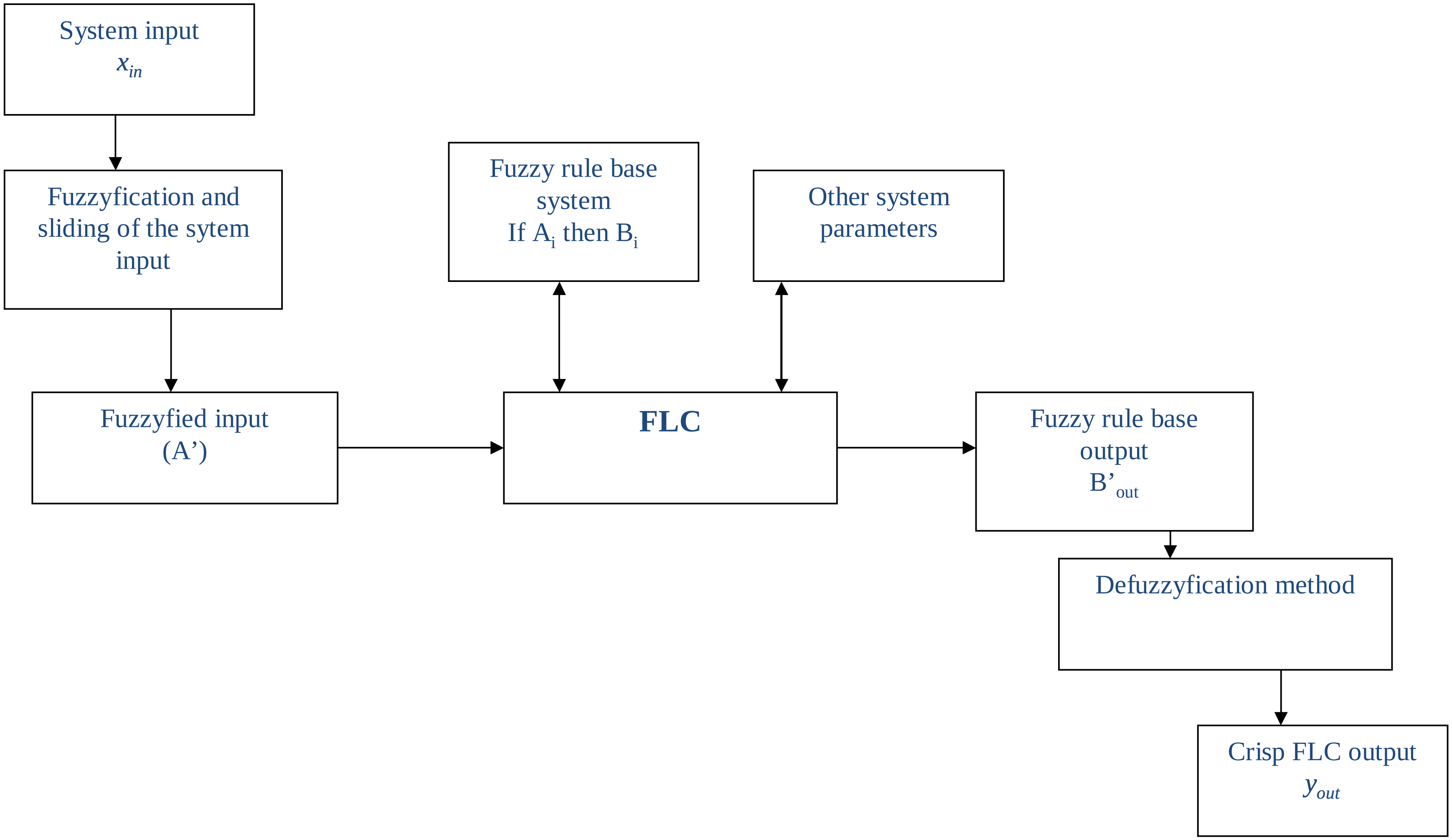
$$(C \circ R)(y) = \sup_{x \in X} T(C(x), R(x, y))$$

for all $y \in Y$.

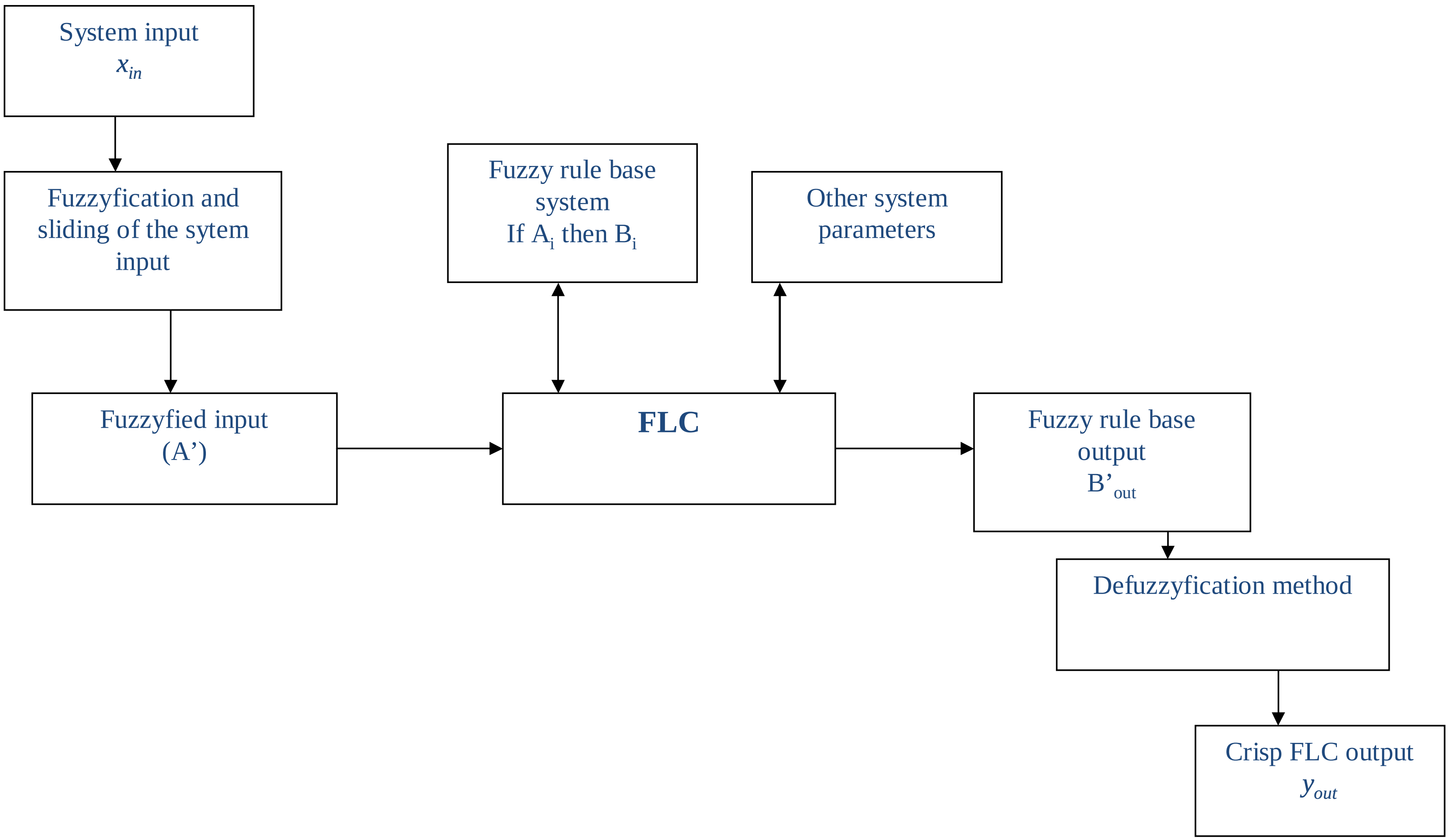
4. Approximate reasoning - közelítő következtetési rendszer

In fuzzy control system the system state is described by a fuzzy rule base, and the relationship between fuzzy rule base system, output of the system and input of the system is modeled by compositional rule of inference.




System input
x_{in}
Fuzzyfication and sliding of the sytem input
Fuzzyfied input
(A’)
Fuzzy rule base system
If A_i then B_i
Other system parameters
FLC
Fuzzy rule base output
B'_{out}
Defuzzyfication method
Crisp FLC output
y_{out}

- Fuzzy systems are ready to be applied in control. However, the following issues have to be taken into account:
- Properties of the resulting control function fC (e.g. smoothness)
- Stability
- Computational complexity (fuzzy controllers often have to be implemented on hardware with limited resources)



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The rule system

Rule1: IF $x = A_1$ THEN $y = B_1$

Rule2: IF $x = A_2$ THEN $y = B_2$

…

Rule n: IF $x = A_n$ THEN $y = B_{n1}$

This is denoted as a *single input, single output* (SISO) system.

- Generalized Modus Ponens (GMP)

$$
\begin{array}{l}
A \quad ® \quad B \\
A' \\
\hline
\qquad B'
\end{array}
$$

The model of the rules

Let be $A \in \mathcal{F}(X)$ and $B \in \mathcal{F}(Y)$, for arbitrary X and Y crisp sets, T a t-norm and

R a fuzzy relation on $X \times Y$. The membership function of the fuzzy subset

$$A \circ_T R$$

of universe Y is given by

$$m_{A \circ_T R}(y) = \sup_{x \in X} \{ T(m_A(x), m_R(x, y)) \},$$

and it is called *compositional rule of inference*.

$$B' = A' \circ_T \left(R_{A \to B} \right)$$

is called a *fuzzy relational equation*

The geometrical interpretation of compositional rule of inference and the interpretations using extensions and projections on fuzzy sets show their effect

$$m_{A' \circ_T R}(y) = \sup_{x \in X} T\left(m_{(A' \times Y) \cap_T R}(x, y) \right) = \sup_{x \in X} T\left(m_{A'}(x), m_R(x, y) \right).$$

Mamdani

The Mamdani type controller is based on Generalized Modus Ponens (GMP)

inference rule, and the rule output is given with a fuzzy set, which is derived from rule consequence, as a cut of them. This cut is the generalized degree of firing level of the rule, considering actual rule base input, and usually it is the supremum of the minimum of the rule premise and rule input (calculating with their membership functions, of course).

If then rules

- In Mamadani-based sources it was suggested to represent an

if *x* is *A* then *y* is *B*

simply as a connection

(for example as a t-norm, *T*(*A*,*B*) or as *min*)

between the so called

rule premise: *x* is *A* and rule consequence: *y* is *B*.

The most significant differences between the models of FLC-s lie in the definition of this connection, relation or implication.

The rule base

Rule1: IF $x = A_1$ THEN $y = B_1$

Rule2: IF $x = A_2$ THEN $y = B_2$

…

Rule n: IF $x = A_n$ THEN $y = B_{n1}$

This is denoted as a *single input, single output* (SISO) system.

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$$m_{A' \circ_T R}(y) = \sup_{x \in X} T\left(m_{(A' \times Y) \cap_T R}(x,y)\right) = \sup_{x \in X} T\left(m_{A'}(x), m_R(x,y)\right).$$

If then rules

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  between the so called

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The most significant differences between the models of FLC-s lie in the definition of this connection, relation or implication.

The Generalized Modus Ponens sees the real influences of the implication or connection choice on the inference mechanisms in fuzzy systems. Usually the general rule consequence for i^{th} rule from a rule system is obtained by

$$B'_i(y)=\sup_{x\in X}(T(A'(x),Imp(A_i(x),B_i(y)))$$

- $B'(y) = \sup_{x\in X} (T(A'(x),T(A(x),B(y)))$
- $B'(y) = T(\sup_{x\in X} (T(A'(x),A(x)),B(y))$
- $B'(y) = \min(\sup_{x\in X} (\min(A'(x),A(x))),B(y))$
- *DOF-degree of firing*
- $B'(y) = \min(DOF,B(y))$

The rule system

AND

Rule1: IF $x = A_1$ THEN $y = B_1$ → B_1'

OR

Rule2: IF $x = A_2$ THEN $y = B_2$ → B_2'

…

OR

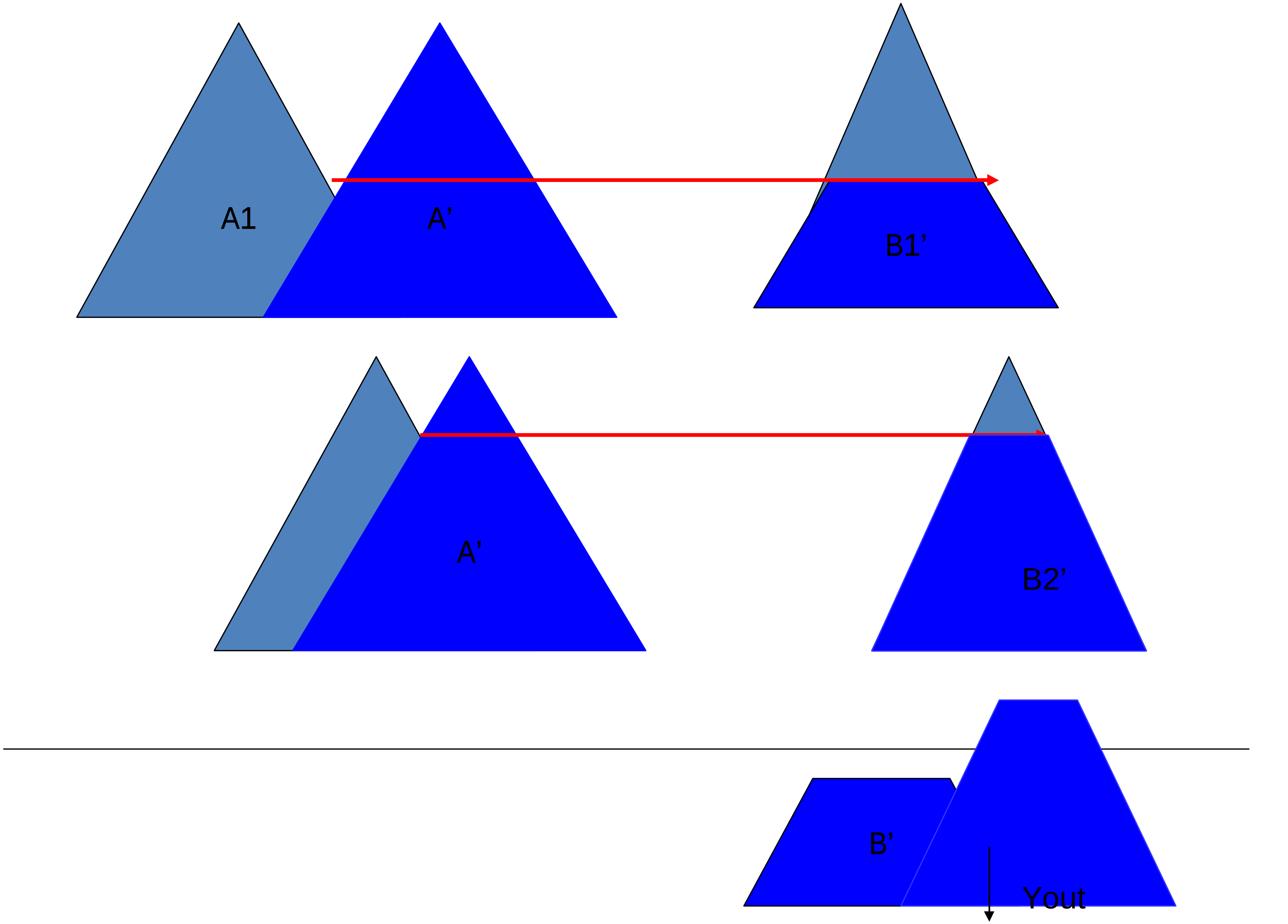
Rule n: IF $x = A_n$ THEN $y = B_{n1}$ → B_3'

↑ AND

$x = A'$

↓

B_{out}


A1
A’
B1’
A’
B2’
B’
Yout

MATLAB exercise

- Build up a simple rule base!
- see http://www.aptronix.com/fide/howfuzzy.htm

Further sources

- http://www.aptronix.com/fide/howfuzzy.htm
- The Fuzzy toolbox, An example

Inger Klein,
Reglerteknik&Kommunikationssystem ISY
Linköpings universitet